

On p -Adic Mathematical Physics

B. Dragovich¹, A. Yu. Khrennikov², S. V. Kozyrev³ and I. V. Volovich³

¹Institute of Physics, Pregrevica 118, 11080 Belgrade, Serbia

²Växjö University, Växjö, Sweden

³Steklov Mathematical Institute, ul. Gubkina 8, Moscow, 119991 Russia

Abstract

A brief review of some selected topics in p -adic mathematical physics is presented.

1 Numbers: Rational, Real, p -Adic

We present a brief review of some selected topics in p -adic mathematical physics. More details can be found in the references below and the other references are mainly contained therein. We hope that this brief introduction to some aspects of p -adic mathematical physics could be helpful for the readers of the first issue of the journal *p -Adic Numbers, Ultrametric Analysis and Applications*.

The notion of numbers is basic not only in mathematics but also in physics and entire science. Most of modern science is based on mathematical analysis over real and complex numbers. However, it is turned out that for exploring complex hierarchical systems it is sometimes more fruitful to use analysis over p -adic numbers and ultrametric spaces. p -Adic numbers (see, e.g. [1]), introduced by Hensel, are widely used in mathematics: in number theory, algebraic geometry, representation theory, algebraic and arithmetical dynamics, and cryptography.

The following view how to do science with numbers has been put forward by Volovich in [2, 3]. Suppose we have a physical or any other system and we

make measurements. To describe results of the measurements, we can always use rationals. To do mathematical analysis, one needs a completion of the field \mathbb{Q} of the rational numbers. According to the Ostrowski theorem there are only two kinds of completions of the rationals. They give real \mathbb{R} or p -adic \mathbb{Q}_p number fields, where p is any prime number with corresponding p -adic norm $|x|_p$, which is non-Archimedean. There is an adelic formula $\prod_p |x|_p = 1$ valid for rational x which expresses the real norm $|x|_\infty$ in terms of p -adic ones. Any p -adic number x can be represented as a series $x = \sum_{i=k}^{\infty} a_i p^i$, where k is an integer and $a_i \in \{0, 1, 2, \dots, p-1\}$ are digits. To build a mathematical model of the system we use real or p -adic numbers or both, depending on the properties of the system [2, 3].

Superanalysis over real and p -adic numbers has been considered by Vladimirov and Volovich [4, 5]. An adelic approach was emphasized by Manin [6].

One can argue that at the very small (Planck) scale the geometry of the spacetime should be non-Archimedean [2, 3, 7]. There should be quantum fluctuations not only of metrics and geometry but even of the number field. Therefore, it was suggested [2] the following *number field invariance principle*: Fundamental physical laws should be invariant under the change of the number field.

One could start from the ring of integers or the Grothendieck schemes. Then rational, real or p -adic numbers should appear through a mechanism of number field symmetry breaking, similar to the Higgs mechanism [8, 9].

Recently (for a review, see [10, 11, 12, 13, 14]) there have been exciting achievements exploring p -adic, adelic and ultrametric structures in various models of physics: from the spacetime geometry at small scale and strings, via spin glasses and other complex systems, to the universe as a whole. There has been also significant progress in non-Archimedean modeling of some biological, cognitive, information and stochastic phenomena.

Ultrametricity seems to be a generic property of complex systems which contain hierarchy. Moreover, there is some evidence towards much more wide applicability of p -adic and non-Archimedean methods to various fields of knowledge. To extend p -adic methods into actual problems in diverse fields of economics, medicine, psychology, sociology, control theory, as well as to many other branches of sciences, is a great challenge and a great opportunity.

2 p -Adic Strings

String theory is a modern unified theory of elementary particles [15].

p -Adic string theories with p -adic valued and also with complex valued amplitudes were suggested by Volovich in [2, 3]. These two possibilities are in correspondence with two forms of the dual string amplitude mentioned in [2]. The first one uses the Veneziano amplitude, which describes scattering of elementary particles, in the form

$$A(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

where a and b are parameters depending on momenta of colliding particles. As a p -adic Veneziano amplitude it was suggested

$$A_p(a, b) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)},$$

where Γ_p is the p -adic valued Morita gamma function. The second one uses the following form of the Veneziano amplitude

$$A(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx.$$

Since the function x^a is a multiplicative character on the real axis we can interpret the Veneziano amplitude as *the convolution of two characters*. In [2] an analogue of the crossing symmetric Veneziano amplitude on the Galois field F_p is also introduced as the convolution of the corresponding complex-valued characters, which is the Jacobi sum,

$$A(a, b) = \sum_{x \in F_p} \chi_a(x) \chi_b(1-x).$$

Freund and Olson [16] introduced an analogue of the crossing symmetric Veneziano amplitude on the \mathbb{Q}_p as the convolution of the corresponding complex-valued characters, which is the Gel'fand-Graev beta function [17],

$$A(a, b) = \int_{\mathbb{Q}_p} \chi_a(x) \chi_b(1-x) dx.$$

Frampton and Okada [18], and Brekke, Freund, Olson and Witten [19] have discovered that string amplitudes given by the above formula and its

generalization can be described by nonlocal effective field theories. Important adelic formulas were considered by Freund and Witten [20], see also [21]. Vladimirov found a unified approach to adelic formulas for the superstring amplitudes using algebraic extensions [22] of the number fields [23] (see also [24, 25, 26]).

Loop corrections to the p -adic string amplitudes are considered in [27, 28]. More information on p -adic string theory see in [29, 30, 12, 10, 31].

Strings, motives and L -functions are discussed by Volovich [32]. String partition function can be expressed as inverse to the Mellin transform of L -function of the Deligne motive,

$$L(s) = \sum_n \tau(n)n^{-s},$$

where $\tau(n)$ is the Ramanujan function.

Motives and quantum fields are discussed by Connes and Marcolli [33]. Motives, algebraic and noncommutative geometry are explored in [34, 35, 36, 37, 38]. Theory of motives is considered by Voevodsky [39].

p -Adic geometry is discussed in [40].

3 p -Adic Field Theories

As a free action in p -adic field theory one can take the following functional

$$S(f) = \int_{\mathbb{Q}_p} f Df dx$$

where $f = f(x)$ is a function $f : \mathbb{Q}_p \rightarrow \mathbb{R}$, dx is the Haar measure and D is the Vladimirov operator or its generalizations [13]. A p -adic analog of the Euclidean quantum field theory was introduced by Kochubei and Sait-Ametov [41].

Renormalizations in p -adic field theory are studied by Missarov [42] and Smirnov [43].

Nonlocal scalar field theory [18, 19] for p -adic strings has occurred to be very instructive toy model [44, 45] for truncated string field theory models [46]. Boundary value problems for homogeneous solutions of nonlinear equations of motion corresponding to the p -adic string [18, 19],

$$e^{\square} \Phi = \Phi^p$$

and its generalizations were explored in [48, 49, 50, 51, 52] (here \square is the d'Alembert operator, the field Φ and its argument are real-valued). A truncated version of the superstring field theory describing non-BPS branes [47]

$$(-\square + 1)e^{\square}\Phi = \Phi^3$$

was investigated in [48, 50, 54, 53].

A generalization of these nonlocal models to the case of curved spacetime has been proposed by Aref'eva [55] and their application to cosmology, in particular, to the inflation and dark energy was initiated. Cosmological applications have been studied in numerous papers [56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67].

Quantization of the Riemann zeta-function and applications to cosmology are considered in [68]. If $\zeta(s)$ is the Riemann zeta-function then the quantum zeta function is the pseudodifferential operator $\zeta(\square)$, i.e. the Riemann zeta-function is the symbol of the pseudodifferential operator $\zeta(\square)$. Quantization of the Langlands program is also indicated.

There is a recent consideration towards an effective Lagrangian for adelic strings with the Riemann zeta function nonlocality [69].

Zeta-functions and many other areas of mathematics are considered by Shai Haran in [70].

A dual connection between p -adic analysis and noncommutative geometry was pointed out by Aref'eva and Volovich [71]. It was found that the Haar measure on quantum group $SU_q(2)$ is equivalent to the Haar measure on p -adic line \mathbb{Q}_p if $q = 1/p$.

Relation between the space of coherent states for free annihilation operators and the space of p -adic distributions was found in [72].

Physics, number theory, and noncommutative geometry are discussed by Connes and Marcolli [73, 74].

The number field invariance principle [2] requires consideration of quantum field theory on an arbitrary number field. Such consideration is attempted in [75].

4 p -Adic and Adelic Quantum Mechanics

A general modern approach to quantum theory is presented in the Varadarajan book [76]. There are several versions of p -adic quantum mechanics. One formulation with the complex valued wave functions given by Vladimirov,

Volovich and Zelenov [77, 78] is based on the triple $\{L^2(\mathbb{Q}_p), W(z), U(t)\}$, where $W(z)$ is a unitary representation of the Heisenberg-Weyl group in the Hilbert space $L^2(\mathbb{Q}_p)$ and $U(t)$ is a unitary dynamics. Representation of canonical commutation relations for p -adic quantum mechanics for finite and infinite dimensional systems were studied by Zelenov [79]. Representation theory of p -adic groups is discussed in [17, 80, 81, 82].

An approach to a unified p -adic and real theory of commutation relations is developed by Zelenov [83]. It is based on the interpretation of the group of functions with values in the field of rational numbers as the experiment data space.

The consequences for particle classification of the hypothesis that space-time geometry is non-archimedean at the Planck scale are explored by Varadarajan [84]. The multiplier groups and universal topological central extensions of the p -adic Poincaré and Galilean groups are determined.

p -Adic Maslov index was constructed by Zelenov [85].

Another formulation [86] uses pseudodifferential operators and spectral theory. The free wave function of p -adic quantum mechanics satisfies a pseudodifferential equation of Schrödinger type. A theory of the Cauchy problem for this equation is developed by Kochubei [87, 13] and Zuniga-Galindo [88]. Matrix valued Schrödinger operator on local fields is considered in [89].

Adelic quantum mechanics, which is a generalization of p -adic and ordinary quantum mechanics as well as their unification, was introduced by Dragovich [90]. It was found that adelic harmonic oscillator is related to the Riemann zeta function [91]. Many adelic physical systems have been considered (as a review, see [92]). As a result of adelic approach and p -adic effects, there is some discreteness of space and time in adelic systems [93]. Distributions on adèles are considered by Dragovich [94], E. M. Radyna and Ya. V. Radyno [95].

p -Adic path (functional) integrals are considered by Parisi [96], Zelenov [97], Varadarajan [98], Smolyanov and Shamarov [99]. Dragovich [90, 100] introduced adelic path integral and elaborated it with Djordjević and Nešić [101]. Using path integral approach, the probability amplitude for one-dimensional p -adic quantum-mechanical systems with quadratic Lagrangians was calculated in the exact form, which is the same as that one in ordinary quantum mechanics [102]. p -Adic Airy integrals are considered in [103].

p -Adic quantum mechanics with p -adic valued wave functions is reviewed below.

5 p -Adic and Adelic Gravity and Cosmology

p -Adic gravity and the wave function of the Universe are considered by Aref'eva, Dragovich, Frampton and Volovich [104]. In particular, p -adic Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} - \Lambda g_{\mu\nu}$$

are explored, where $g_{\mu\nu}$ is p -adic valued gravitational field. Summation on algebraic varieties and adelic products in quantum gravity are investigated.

Adelic quantum cosmology, as an application of adelic quantum mechanics to minisuperspace cosmological models of the very early universe, is initiated by Dragovich [100, 105] and developments are presented in [106, 107]. It is illustrated by a few cosmological models, and some discreteness of the minisuperspace and the cosmological constant are found.

As was mentioned above, nonlocal scalar field theories for p -adic strings have occurred to be interesting in cosmology, in particular, in the context of their relation with nonlocal string field inspired models [55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67].

6 p -Adic Stochastic Processes

The p -adic diffusion (heat) equation

$$\frac{\partial f(x, t)}{\partial t} + D_x^\alpha f(x, t) = 0$$

was suggested in [10], and its mathematical properties and properties of the associated stochastic processes were studied (see also [108, 109, 110, 98]). Here $f = f(x, t)$ is a real valued function of the real time t and the p -adic coordinate x . D_x^α is the Vladimirov operator.

p -Adic Brownian motion is explored in [111, 112, 113], see also [114]. Various classes of p -adic stochastic processes are investigated by Albeverio and Karwowski [115], Kochubei [13], Yasuda [116], Albeverio and Belopolskaya [117]; see [13] for further references. For the most recent results on p -adic stochastic integrals and stochastic differential equations see [118].

Kaneko [119] showed a relationship between Besov space and potential space, and pointed out a probabilistic significance of the relationship in terms of fractal analysis.

7 Vladimirov operator

The Vladimirov operator [120, 10] of p -adic fractional differentiation is defined as

$$D^\alpha f(x) = \int_{\mathbb{Q}_p} \chi(-kx) |k|_p^\alpha dk \tilde{f}(k) dk, \quad \tilde{f}(k) = \int_{\mathbb{Q}_p} \chi(kx) f(x) dx.$$

Here $f(x)$ is a complex-valued function of p -adic argument x and χ is the additive character: $\chi(x) = \exp(2\pi i \{x\})$, $\{x\}$ is the fractional part of x .

For $\alpha > 0$ the Vladimirov operator has the following integral representation

$$D^\alpha f(x) = \frac{1}{\Gamma_p(-\alpha)} \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{|x - y|_p^{1+\alpha}} dy,$$

with the constant

$$\Gamma_p(-\alpha) = \frac{p^\alpha - 1}{1 - p^{-1-\alpha}}.$$

There is a well-developed theory of the Vladimirov operator and related constructions (spectral properties, operators on bounded regions, analogs of elliptic and parabolic equations, a wave-type equation etc); see section on wavelets and [10, 13, 121, 122, 123].

8 Dynamics and Evolution of Complex Biological Systems

An idea of using ultrametric spaces to describe the states of complex biological systems, naturally possess a hierarchical organization, has been sound more than once as from the middle of the 1980th by G. Frauenfelder, G. Parisi, D. Stain, and the others, see [124]. In protein physics, it is regarded as one of the most profound ideas put forward to explain the nature of distinctive life attributes, since it proposes, in a wide extend, the existence of very peculiar order inherent the information and functional carriers in biology.

In earlier theoretical examination of this idea, some models of ultrametric random walk were proposed, but they were confronted with difficulties in applications to the protein fluctuation dynamics, in particular, to the large body of data obtained in the experiments on ligand-rebinding kinetics of myoglobin and spectral diffusion in deeply frozen proteins.

Realization of this task has been provided for last years by V. Avetisov in collaboration with A. Bikulov, S. Kozyrev, V. Osipov, and A. Zubarev [125, 126, 127, 128, 129]. It was shown that, in spite of extreme complexity of the protein energy landscape, p -adic diffusion equation introduced in [10] offers a surprisingly simple, accurate, and universal description of the protein fluctuation dynamics on extremely large range of scales from cryogenic up to room temperatures:

$$\frac{\partial f(x, t)}{\partial t} + D_x^\alpha f(x, t) = 0, \quad \alpha \sim \frac{1}{T}.$$

Here t is the real time and the p -adic coordinate x describes the “tree of basins” which corresponds to the conformational state of the protein, T is the temperature. This equation was used to describe two drastically different types of experiments — on rebinding of CO to myoglobin and spectral diffusion of proteins.

These applications of p -adic diffusion equation to the protein dynamics highlight very important protein attribute, namely, the fact that protein dynamic states and protein energy landscape, being both extremely complex, exhibit the hierarchical self-similarity.

On the opposite side of biological complexity, e.g. in modeling of optimization selection of complex biological entities over combinatorial large evolutionary spaces, the p -adic stochastic processes have recently been recognized as a useful tool too. Though the fact of natural ultrametric (taxonomic) relationships in biology, the ultrametric representation of biological realm has not been reflected by models.

Such an evolutionary model, based on p -adic diffusion equation, has recently been proposed by V. Avetisov and Yu. Zhuravlev [130]. It is interesting, that the model, being suggested to describe the evolution of complex biological entities, cast light on the basic point of the prebiotic evolutionary concepts known as the “error catastrophe”. It was found that the prebiotic evolution can be getting beyond the continuity principle of Darwinian evolutionary paradigm.

9 Quantization with p -Adic Valued Wave Functions

The first step toward quantum mechanics with wave functions valued in non-Archimedean fields (and even superalgebras) was done by Vladimirov and Volovich [4, 5], see also [3]. This approach was elaborated by Khrennikov in series of papers and books [131, 132, 133, 134], [11] and extended in collaboration with Cianci and Albeverio [135, 136, 137, 147, 148]. Here we present essentials of this theory. The basic objects of this theory are the p -adic Hilbert space and symmetric operators acting in this space [149, 150]. Vectors of the p -adic Hilbert space which are normalized with respect to inner product represent quantum states. p -Adic valued quantum theory suffers of the absence of a “good spectral theorem”. At the same time this theory is essentially simpler (mathematically), since *operators of position and momentum are bounded*. Thus the canonical commutation relations can be represented by bounded operators¹. It is impossible in the complex case. Representations of groups in Hilbert spaces are cornerstones of quantum mechanics. In [135] Albeverio and Khrennikov constructed a representation of the Weyl–Heisenberg group in the p -adic Hilbert space. Spectra of the position and momentum operators were studied in [136], [137].

Theory of p -adic valued functions is exposed in the Schikhof book [138]. Spectral theory in ultrametric Banach algebras and the ultrametric functional calculus are considered by Escassut [139], analytic functions on infraconnected sets are studied in [140]. Theory of meromorphic functions over non-Archimedean fields is presented by Pei-Chu Hu and Chung-Chun Yang [141]. Differential equations for p -adic valued functions were studied by many authors, see [142, 143]. p -Adic summability of perturbation series is also investigated, see [144, 145, 146].

10 \mathbb{Q}_p -valued Probability

This quantum model induces \mathbb{Q}_p -valued “probabilities”. Surprisingly we can proceed in the rigorous probabilistic framework even in such unusual situation – including limit theorems and theory of randomness, see Khrennikov

¹It was discovered by Albeverio and Khrennikov [135] and later and independently by Kochubei [151] (with a different construction of the representation), see also Keller, Ochsenius, and Schikhof [152].

[134, 11, 147, 148]. The starting point of such a generalized probabilistic model was extension of von Mises' frequency probability theory to p -adic topologies: frequencies of realizations should stabilize not with respect to the ordinary real topology, but one of p -adic topologies. We emphasize that relative frequency is always a rational number: $\nu_n(\alpha) = n(\alpha)/N$, where N is the total number measurements and n is the number of realizations favorable to the fixed result α . In the simplest case $\alpha = 0, 1$. Thus the p -adic version of frequency probability theory describes a new class of random sequences, a new type of randomness.

11 Applications to Cognitive Science and Psychology

The idea of encoding mental states by numbers has very long history – Plato, Aristotle, Leibnitz, and the others. A new realization of this idea was provided by Khrennikov who used rings of m -adic numbers \mathbb{Z}_m to encode states of the brain, see [147]. This approach was developed in collaboration with Albeverio, Kloeden, Tirozzi, Gundlach, Dubischar, Steinkamp [153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163], [148]. The model of m -adic (and more general ultrametric) *mental space* was applied to cognitive science and psychology, including modeling of coupling between unconscious and conscious flows of information in the brain. Coupling with neurophysiology was studied in [162, 163] – the process of “production” of m -adic mental space by neuronal trees. This model can be used for description of arbitrary flows of information on the basis of m -adic *information space*. The main distinguishing feature of this approach is a possibility to encode hierarchical structures which are present in information by using ultrametric topologies, treelike structures. Ultrametric algorithmic information is considered by Murtagh [164].

12 Applications to Image Analysis

Practically all images have (often hidden) hierarchical structures. These structures can be represented by using m -adic information spaces. In some cases the m -adic representation of images essentially simplifies analysis, in particular, *image recognition*. One can ignore details which belong to lower

levels of the m -adic hierarchy of an image. More effective algorithms can be developed starting with the m -adic encoding of information. This approach was developed by Benois-Pineau, Khrennikov, Kotovich, and Borzistaya [165, 166, 167].

13 p -Adic Wavelets

p -Adic wavelet theory was initiated by Kozyrev [168]. Since in the p -adic case it is not possible to use for construction of wavelet basis translations by elements of \mathbb{Z} , it was proposed to use instead translations by elements of the factor group $\mathbb{Q}_p/\mathbb{Z}_p$. An example of p -adic wavelet was introduced in the form of the product of the character and the characteristic function Ω of the unit ball:

$$\psi(x) = \chi(p^{-1}x)\Omega(|x|_p).$$

The orthonormal basis $\{\psi_{\gamma nj}\}$ of p -adic wavelets in $L^2(\mathbb{Q}_p)$ was constructed [168] by translations and dilations of ψ :

$$\psi_{\gamma nj}(x) = p^{-\frac{\gamma}{2}}\chi(p^{\gamma-1}j(x - p^{-\gamma}n))\Omega(|p^\gamma x - n|_p),$$

$$\gamma \in \mathbb{Z}, \quad n = \sum_{l=k}^{-1} n_l p^l \in \mathbb{Q}_p/\mathbb{Z}_p, \quad n_l = 0, \dots, p-1, \quad j = 1, \dots, p-1.$$

In p -adic analysis wavelet bases are related to the spectral theory of pseudodifferential operators. In particular, the above p -adic wavelets are eigenvectors of the Vladimirov operator:

$$D^\alpha \psi_{\gamma nj} = p^{\alpha(1-\gamma)} \psi_{\gamma nj}.$$

Spectra of more general operators were studied in [169].

Moreover, the orbit of the wavelet ψ with respect to the action of the p -adic affine group

$$G(a, b)f(x) = |a|_p^{-\frac{1}{2}} f\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{Q}_p, \quad a \neq 0,$$

gives the set of products of p -adic wavelets from the basis $\{\psi_{\gamma nj}\}$ and roots of one of the degree p . In general [170], the orbit of generic complex-valued locally constant mean zero compactly supported function gives a frame of p -adic wavelets.

Relation between real and p -adic wavelets is given by the p -adic change of variable: the map

$$\rho : \mathbb{Q}_p \rightarrow \mathbb{R}_+, \quad \rho : \sum_{i=\gamma}^{\infty} x_i p^i \mapsto \sum_{i=\gamma}^{\infty} x_i p^{-i-1}, \quad x_i = 0, \dots, p-1, \quad \gamma \in \mathbb{Z}$$

maps (for $p = 2$) the basis $\{\psi_{\gamma n_j}\}$ of p -adic wavelets onto the basis of Haar wavelets on the positive half-line.

Important contributions in p -adic wavelet theory were done by Albeverio, Benedetto, Khrennikov, Shelkovich, Skopina [171, 172, 173, 174, 175, 176]. p -Adic Tauberian and Shannon-Kotelnikov theorems were investigated by Khrennikov and Shelkovich [177].

14 Analysis on General Ultrametric Spaces

The analysis of wavelets and pseudodifferential operators on general locally compact ultrametric spaces was developed in [178, 179, 180, 14]. A pseudodifferential operator on ultrametric space X is defined by Kozyrev as the following integral operator:

$$Tf(x) = \int_X T(\text{sup}(x, y))(f(x) - f(y)) d\nu(y)$$

where ν is a Borel measure and some integrability condition for the integration kernel is satisfied. The $\text{sup}(x, y)$ is the minimal ball in X which contains the both points x and y , i.e. the integration kernel T is a locally constant (for $x \neq y$) function with the described domains of local constancy.

Ultrametric wavelet bases $\{\Psi_{I_j}\}$ were introduced and it was found that ultrametric wavelets are eigenvectors of ultrametric pseudodifferential operators:

$$T\Psi_{I_j} = \lambda_I \Psi_{I_j}.$$

Let us note that locally compact ultrametric spaces under consideration are completely general and do not possess any group structure.

Analysis on p -adic infinite-dimensional spaces was developed by Kochubei, Kaneko and Yasuda [181, 13, 182, 183, 184].

Another important class of ultrametric spaces consists of locally compact fields of positive characteristic. Analysis over such fields has both common and different features with p -adic analysis. See the book of Kochubei [185] for the details.

15 Cascade Models of Turbulence

One of the problems of fully developed turbulence is the energy-cascading process. This process is characterized by large hierarchy of scales involved and is known as the Richardson cascade. The main idea of the Fischenko and Zelenov paper [186] was to consider the equation of cascade model not in coordinate space, but in an ultrametric space connected with hierarchy of turbulence eddies. This idea leads to a rather simple nonlinear integral equation for the velocity field.

Properties of fully developed turbulence such as multifractal behavior of energy dissipation or Kolmogorov's 1/3 behavior are obtained by analysis of solutions of the nonlinear equation over p -adic numbers. So, p -adic numbers provide a natural and systematic approach to cascade models of turbulence.

A modification of nonlinear ultrametric integral equation of [186] was investigated in [187] by Kozyrev. It was found [187] that, using the ultrametric wavelet analysis, we get a family of exact solutions for this modified nonlinear equation. Moreover, for this equation an exact solution of an arbitrary Cauchy problem with the initial condition in the space of locally constant mean zero compactly supported functions was constructed.

16 Disordered Systems and Spin Glasses

Spin glasses (disordered magnetics) are typical examples of disordered systems. Order parameter for a spin glass in the replica symmetry breaking approach is described by the Parisi matrix — some special hierarchical block matrix. It was found that [188] the structure of correlation functions for spin glasses are related to ultrametricity.

Avetisov, Bikulov, Kozyrev [125] and Parisi, Sourlas [189] found that the Parisi matrix possesses the p -adic parametrization, namely matrix elements of this matrix after some natural enumeration of the lines and columns of the matrix can be expressed as the real valued function of the p -adic norm of difference of the indices

$$Q_{ab} = q(|a - b|_p).$$

This allows to express the correlation functions of the spin glass in the state with broken replica symmetry in the form of some p -adic integrals.

Also more general replica solutions related to general locally compact ultrametric spaces were obtained [190, 191, 192]. The p -adic Potts model is considered in [193, 194].

17 p -Adic Dynamical Systems

The theory of p -adic dynamical systems [195] is an intensively developing discipline on the boundary between various mathematical theories – dynamical systems, number theory, algebraic geometry, non-Archimedean analysis – and having numerous applications – theoretical physics, cognitive science, cryptography, computer science, automata theory, genetics, numerical analysis and image analysis.

One of the sources of theory of p -adic dynamical systems was dynamics in rings of $\bmod p^n$ -residue classes. We can mention investigations of W. Narkiewicz, A. Batra, P. Morton and P. Patel, J. Silverman and G. Call, D.K. Arrowsmith, F. Vivaldi and Hatjispyros, J. Lubin, T. Pezda, H-C. Li, L. Hsia, see, e.g., [197], [196] and also books [195], [198] for detailed bibliography. Another flow was induced in algebraic geometry. This (algebraic geometric) dynamical flow began with article of M. Herman and J.C. Yoccoz [199] on the problem of small divisors in non-Archimedean fields. It seems that this was the first publication on non-Archimedean dynamics. In further development of this dynamical flow the crucial role was played by J. Silverman, R. Benedetto, J. Rivera-Letelier, C. Favre, J-P. Bézivin, see Silverman's book [200] for bibliography.

Another flow towards algebraic dynamics has p -adic theoretical physics as its source. One of the authors of this review used this pathway towards p -adic dynamical systems, from study of quantum models with \mathbb{Q}_p -valued functions. As a result, a research group on non-Archimedean dynamics was created at the Växjö University, Sweden: Andrei Khrennikov, Karl-Olof Lindahl, Marcus Nilsson, Robert Nyqvist, and Per-Anders Svensson [195].

We point out recent publications of V. Arnold, e.g., [201], devoted to chaotic aspects of arithmetic dynamics closely coupled to the problem of turbulence. Some adelic aspects of linear fractional dynamical systems are considered in [202].

Finally, we point out a flow towards algebraic dynamics which is extremely important for applications to computer science, cryptography, and numerical analysis, especially in connection with pseudorandom numbers and uniform

distribution of sequences. This flow arose in 1992 starting with works of Anashin [203, 204], followed by a series of his works on p -adic ergodicity as well as on above applications. It worth mention here one of the most recent papers from these, [205] that contains a solution of a problem on perturbed monomial mappings on p -adic spheres – the problem was put by A.Khrennikov, e.g., [195], see [198] for general presentation of p -adic ergodic theory.

18 p -Adic Models of the Genetic Code

Recently, a new interesting application of m -adic information space was found in the domain of genetics, ($m = 2, 4, 5$ depending on a model); see B. Dragovich and A. Dragovich [206], A. Khrennikov [207], M. Pitkänen [208], A. Khrennikov and S. Kozyrev [211].

The relation between 64 codons, which are building blocks of genes, and 20 amino acids, which are building blocks of proteins, is known as the genetic code, which is degenerate. Since G. Gamow, there has been a problem of theoretical foundation of experimentally known (only about 16) codes in all living organisms. The central point of p -adic approach to the genetic code is identification $C=1, A=2, T=3, G= 4$, where C, A, T, G are nucleotides in DNA and 1, 2, 3, 4 are digits in 5-adic representation of codons, which are trinucleotides. Using p -adic distances between codons, it was shown that degeneration of the vertebral mitochondrial code has p -adic structure, and all other codes can be regarded as slight modifications of this one. Details of this approach can be found in [209] of the present volume and [210].

The following p -adic model of the genetic (amino acid) code was proposed in [211]. It was shown that after some p -adic parametrization of the space of codons the degeneracy of the amino acid code is equivalent to the local constancy of the map of a p -adic argument. In two-dimensional 2-adic parametrization of [211] we get the following table of amino acids on the

2-adic plane of codons:

$\frac{\text{Lys}}{\text{Asn}}$	$\frac{\text{Glu}}{\text{Asp}}$	$\frac{\text{Ter}}{\text{Ser}}$	Gly
$\frac{\text{Ter}}{\text{Tyr}}$	$\frac{\text{Gln}}{\text{His}}$	$\frac{\text{Trp}}{\text{Cys}}$	Arg
$\frac{\text{Met}}{\text{Ile}}$	Val	Thr	Ala
$\frac{\text{Leu}}{\text{Phe}}$	Leu	Ser	Pro

19 Applications to Economics, Finance, Data Mining

Possible applications of p -adic analysis in economics, finance, business connections, including a p -adic version of the Black-Scholes equation [213], are discussed in [212, 213, 214, 215].

Application of p -adic analysis in data analysis and data mining is explored by Murtagh [216].

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